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On A Simple Construction Of Triangle Centers X(8), X(197), X(K) & X(594)

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Abstract

In this article, we provide a new method for constructing the Nagel Point{X(8)} , Cevian quotient of Symmedian Point, Nagel Point {X(197)} and the isogonal conjugate of 1st Hatzipolakis-Yiu Point{X(594)}. In addition, we also establish some collinearity and concurrence.

Keywords: triangle centers, Nagel point, Carnot's theorem, collinearity and concurrence.

Introduction

In the literature of Encyclopedia of Triangle Centers **[1]**, there is a list of over 2000 triangle centers. Among those X(8), X(197) , X(K) and X(594) are four such triangle centers. In this note, we devote our study for the construction of these points and their related coincidence.

In article **[2],** *Larry Hoehn* gives another way to construct the Nagel point using only the incircle and not the excircles. In this article we deal with a simple and an elegant construction of these four points which reveals a new characterization of X(8), X(197), X(K), X(594) and their unexpected coincidence.

Terminology

Triangle center	Its name
X(8)	Nagel Point
X(197)	Cevian Quotient of Symmedian Point,Nagel Point
X(K)	Isogonal Conjugate of X(5019), Isotomic Conjugate of X(940), The Polar Conjugate of X(4185), The Trilinear Pole Of Line X(523)X(4391)
X(594)	Isogonal Conjugate of 1st Hatzipolakis-Yiu Point

Construction

Given a triangle *ABC*, let *D*, *E*, *F* be the midpoints of the sides BC, CA and AB, construct a circle O_A with *A* as center and *AD* as radius which intersects the line drawn through D and parallel to the internal angular bisector of angle *A* at L_A (it is clear that the point L_A is the reflection of *D* with respect to the external angular bisector of angle *A*). Similarly, define the points L_B and L_C. Consider the points N_A, N_B and N_C as

$N_A = BL_C \cap CL_B$, $N_B = AL_C \cap CL_A$ and $N_C = BL_A \cap AL_B$ then

(I) ***The line segments AN_A, BN_B and CN_C concur at X(594)***
(II) ***The line segments L_AN_A, L_BN_B and L_CN_C concur at X(8)***

Consider the points N'_A, N'_B, N'_C, L'_A, L'_B, L'_C, M'_A, M'_B and M'_C as $N'_A = AN_A \cap BC$, $N'_B = BN_B \cap CA$, $N'_C = CN_C \cap AB$, $L'_A = AX(8) \cap BC$, $L'_B = BX(8) \cap CA$, $L'_C = CX(8) \cap AB$, $M'_A = L_A N_A \cap BC$, $M'_B = L_B N_B \cap CA$ and $M'_C = L_C N_C \cap AB$ then

(III) ***The line segments*** AM'_A, BM'_B ***and*** CM'_C ***are concur at X(K)***

(IV) ***The lines*** $N_A L'_A$, $N_B L'_B$ ***and*** $N_C L'_C$ ***are concur at X(197)***

(V) ***There exists a conic through any two triads out of three triads (D, E, F),*** (N'_A, N'_B, N'_C) ***and*** (L'_A, L'_B, L'_C).

In order to prove (1), (II), (III), (IV) and (V), we will make use of barycentric coordinates. If a triangle ABC has side lengths BC = a, CA = b, AB = c then A = (1 : 0 : 0), B = (0 : 1 : 0), C = (0 : 0 : 1), D = (0 : 1 : 1), E = (1 : 0 : 1) and F = (1 : 1 : 0) in homogeneous barycentric coordinates with reference to ABC **[4].**

Preposition 1

The equation of line joining of two points with coordinates $(x_1 : y_1 : z_1)$ *and*

$(x_2 : y_2 : z_2)$ *is* $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$ *or* $x(y_1z_2 - y_2z_1) + y(z_1x_2 - z_2x_1) + z(x_1y_2 - x_2y_1) = 0$

Preposition 2

The intersection of the two lines $p_1x + q_1y + r_1z = 0$ *and* $p_2x + q_2y + r_2z = 0$ *is the point* $(q_1r_2 - q_2r_1 : r_1p_2 - r_2p_1 : p_1q_2 - p_2q_1)$.

Preposition 3

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$, *are concurrent if and only if*

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

Preposition 4

The barycentric coordinate of the points which are the reflection of an arbitrary point $P(u : v : w)$ *with respect to the external angular bisector of angle A is*

$P_A = (c^2 v + b^2 w + b c (u + v + w) : -b^2 w : -c^2 v)$,

with respect to the external angular bisector of angle B is

$P_B = (-a^2 w : c^2 u + a^2 w + a c (u + v + w) : -c^2 u)$,

and with respect to the external angular bisector of angle C is

$P_C = (-a^2 v : -b^2 u : b^2 u + a^2 v + a b (u + v + w))$ **[4].**

Corllory

By replacing u = 0, v = w = 1 then $P_A = L_A = ((b+c)^2 : -b^2 : -c^2)$

By replacing v = 0, u = w = 1 then $P_B = L_B = (-a^2 : (a+c)^2 : -c^2)$

By replacing w = 0, u = v = 1 then $P_C = L_C = (-a^2 : -b^2 : (a+b)^2)$

Note

1. The triangles ABC and $P_AP_BP_C$ are perspective with perpector of the isogonal conjugate of P **[4]**.
2. The excentral triangle and $P_AP_BP_C$ are perspective if and only if P lies on the Neuberg cubic of excentral triangle **[4].**

Using the prepositions listed above we list out the barycentric coordinates of the specified points and equation of the lines in barycentric system (we use standard notations such as a = BC, b = CA, c = AB and s = semi perimeter, R= circum radius, r = inradius, Δ= area).

Table 1. Point and its Barycentric Coordinates

Point	barycentric coordinates	Point	barycentric coordinates
L_A	$((b+c)^2 : -b^2 : -c^2)$	L'_A	$(0 : (s-b) : (s-c))$
L_B	$(-a^2 : (a+c)^2 : -c^2)$	L'_B	$((s-a) :0: (s-c))$
L_C	$(-a^2 : -b^2 : (a+b)^2)$	L'_C	$((s-a): (s-b) :0)$
N_A	$(-a^2 : (c+a)^2 : (a+b)^2)$	N'_A	$(0 : -(c+a)^2: -(a+b)^2)$
N_B	$((b+c)^2 : -b^2 : (a+b)^2)$	N'_B	$(-(b+c)^2 : 0: -(a+b)^2)$
N_C	$((b+c)^2 : (c+a)^2 : -c^2)$	N'_C	$(-(c+b)^2 : -(c+a)^2 : 0)$

Point	barycentric coordinates
M'_A	$(0 : sc^2+abc: sb^2+abc) = (0 : c^2+4Rr : b^2+4Rr)$
M'_B	$(c^2+4Rr : 0 : a^2+4Rr)$
M'_C	$(b^2+4Rr : a^2+4Rr : 0)$

Point	barycentric coordinates
X(8)	$((s-a) : (s-b) : (s-c)) = (\cot A/2 : \cot B/2 : \cot C/2)$
X(197)	$(a^2 [s^2 (b^2 + c^2 - a^2) – s (bc(b + c) + ca(c - a) + ab(b - a) + abc(b + c - a)] : b^2 [s^2 (c^2 + a^2 - b^2) – s (ca(c + a) + ab(a - b) + bc(c - b) + abc(c + a - b)] : c^2 [s^2 (a^2 + b^2 - c^2) – s (ab(a + b) + bc(b - c) + ca(a - c) + abc(a + b - c)])$ $= af(A,B,C) : bf(B,C,A) : cf(C,A,B)$ Where $f(A,B,C) = = a[-a^2\tan A/2 + b^2\tan B/2 + c^2\tan C/2]$
X(594)	$((c+b)^2 : (c+a)^2 : (a+b)^2)$
X(K)	$((b^2+4Rr)(c^2+4Rr) : (c^2+4Rr)(a^2+4Rr): (a^2+4Rr)(b^2+4Rr))$

Table 2. Line and its Barycentric Equation

Line	Equation	Line	Equation	Line	Equation
AL_B	$yc^2 + z(c+a)^2 = 0$	BL_A	$xc^2 + z(b+c)^2 = 0$	CL_B	$x(c+a)^2 + ya^2 = 0$
AL_C	$y(a+b)^2 + zb^2 = 0$	BL_C	$x(a+b)^2 + za^2 = 0$	CL_A	$xb^2 + y(b+c)^2 = 0$

Line	Equation
AN_A	$y(a+b)^2 - z(c+a)^2 = 0$
BN_B	$x(a+b)^2 - z(b+c)^2 = 0$
CN_C	$x(c+a)^2 - y(b+c)^2 = 0$

Line	Equation
$AX(8)$	$y(s-c) - z(s-b) = 0$
$BX(8)$	$x(s-c) - z(s-a) = 0$
$CX(8)$	$x(s-b) - y(s-a) = 0$

Line	Equation
L_AN_A	$xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0$
L_BN_B	$yca(c-a) - s[c^2(y+z) - a^2(y+x)] - abc(z-x) = 0$
L_CN_C	$zab(a-b) - s[a^2(z+x) - b^2(z+y)] - abc(x-y) = 0$

Line	Equation
AM'_A	$y(b^2+4Rr) - z(c^2+4Rr) = 0$
BM'_B	$x(a^2+4Rr) - z(c^2+4Rr) = 0$
CM'_C	$x(a^2+4Rr) - y(b^2+4Rr) = 0$

Line	Equation
$N_AL'_A$	$a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0$
$N_BL'_B$	$b^2[z(s-a)-x(s-c)] + y(c-a)[s(c+a)-ca] = 0$
$N_CL'_C$	$c^2[x(s-b)-y(s-a)] + z(a-b)[s(a+b)-ab] = 0$

Theorem-1

The line segments AN_A, BN_B and CN_C are concur at X(594).

Proof

Using table 2, it is easy to check that the point X(594){$((c+b)^2 : (c+a)^2 : (a+b)^2)$} satisfies the lines AN_A {$y(a+b)^2 - z(c+a)^2 = 0$}, BN_B {$x(a+b)^2 - z(b+c)^2 = 0$} and CN_C{ $x(c+a)^2 - y(b+c)^2 = 0$}.

Hence the lines AN_A, BN_B and CN_C are concurrent at X(594).*(see figure -1)*

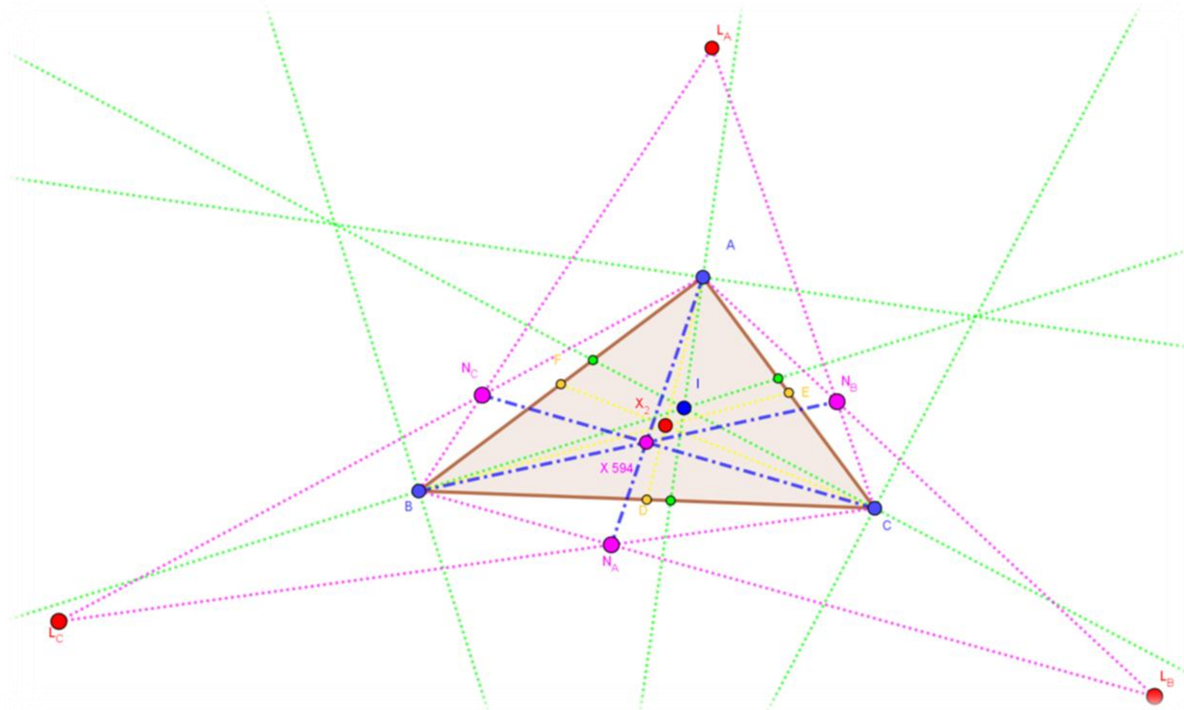

Figure 1.

Theorem-2

The line segments L_AN_A, L_BN_B and L_CN_C are concur at X(8).

Proof

Using table 2, it is easy to check that the point X(8){((s-a) : (s-b) : (s-c))} satisfies the lines L_AN_A { $xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0$ }, L_BN_B { $yca(c-a) - s[c^2(y+z) - a^2(y+x)] - abc(z-x) = 0$} and L_CN_C { $zab(a-b) - s[a^2(z+x) - b^2(z+y)] - abc(x-y) = 0$ }.

For example

Consider $xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = 0$ ……(1)

Replace $x = (s-a)$, $y=(s-b)$, $z = (s-c)$, $x+y = c$, $x+z = b$ and $y-z = c-b$

$$xbc(b-c) - s[b^2(x+y) - c^2(x+z)] - abc(y-z) = bc(s-a)(b-c) - s[b^2c - c^2b]-abc(c-b)$$

$$= bc(s-a)(b-c) - bc(s-a)(b-c)=0$$

Hence (1) is satisfied by the point X(8){((s-a) : (s-b) : (s-c))}

That is the line L_AN_A contains X(8).

In the similar manner we can verify that L_BN_B and L_CN_C also contains X(8).

So the lines L_AN_A, L_BN_B and L_CN_C are concur at **X(8).** *(see figure -2)*

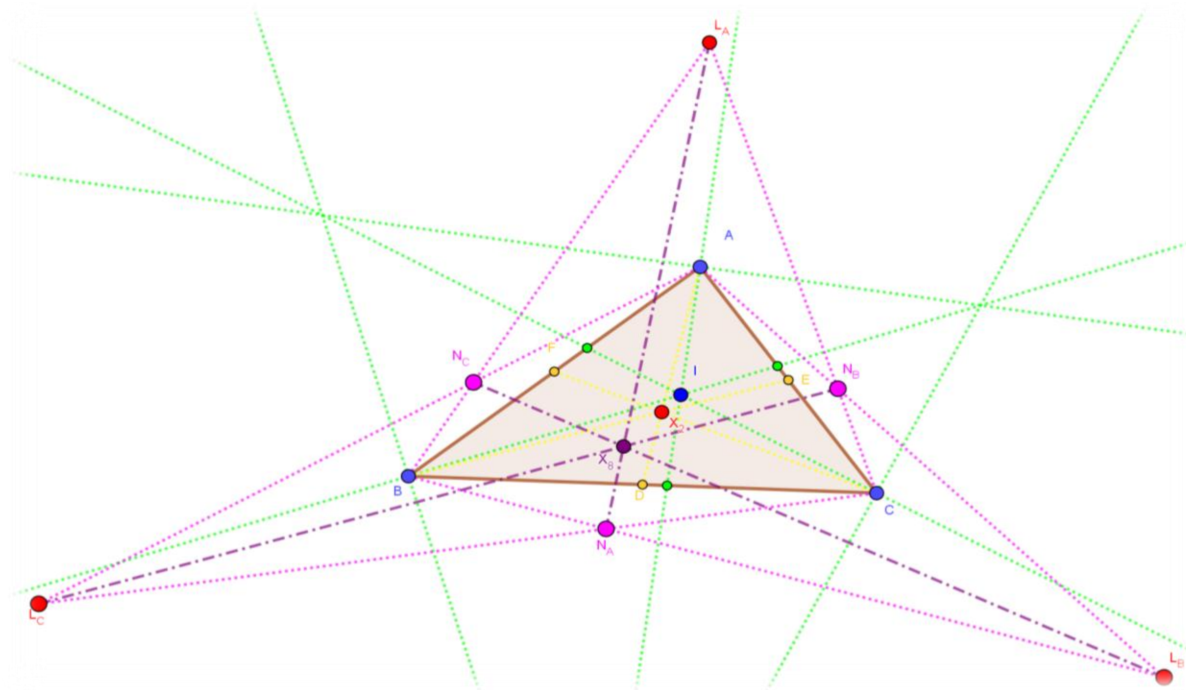

Figure 2.

Theorem – 3

The line segments AM'_A, BM'_B *and* CM'_C *are concur at X(K)*

Proof

It is clear using table 2, the equations of lines AM'_A, BM'_B *and* CM'_C are

$y(b^2+4Rr) - z(c^2+4Rr) = 0$, $x(a^2+4Rr) - z(c^2+4Rr) = 0$ and $x(a^2+4Rr) - y(b^2+4Rr) = 0$.

Now using Preposition 3, to prove that these lines are concurrent, it is enough to prove that

$$\begin{vmatrix} 0 & \left(b^2+4Rr\right) & -(c^2+4Rr) \\ \left(a^2+4Rr\right) & 0 & -\left(c^2+4Rr\right) \\ \left(a^2+4Rr\right) & -\left(b^2+4Rr\right) & 0 \end{vmatrix} = 0$$

which is true.

They concur at a point X(K) whose barycentric coordinate is

$((b^2+4Rr)(c^2+4Rr) : (c^2+4Rr)(a^2+4Rr):(a^2+4Rr)(b^2+4Rr))$. *(see figure -3)*

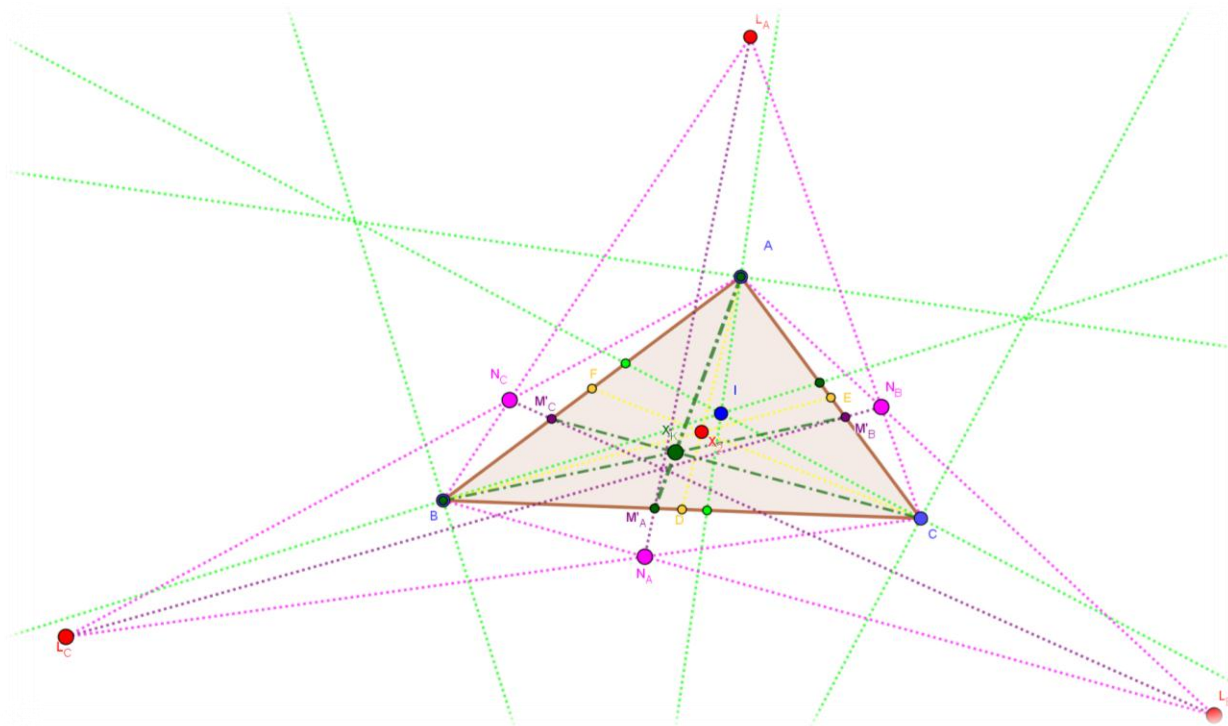

Figure 3.

Note

The point X(K) is noted as the isogonal conjugate of X(5019) by *Francisco Javier* **[5]** and as the isotomic conjugate of X(940), the polar conjugate of X(4185) and the trilinear pole of line X(523), X(4391) by *Randy* **[6].**

Theorem 4

The lines $N_A L'_A$**,** $N_B L'_B$ ***and*** $N_C L'_C$ ***are concurrent at X(197)***

Proof

Using table 2, it is easy to check that the point X(197) {af(A,B,C) : bf(B,C,A) : cf(C,A,B)} satisfies the lines

$N_A L'_A$ {a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0},

$N_B L'_B$ {b^2[z(s-a)-x(s-c)] + y(c-a)[s(c+a)-ca] = 0}

and $N_C L'_C$ {c^2[x(s-b)-y(s-a)] + z(a-b)[s(a+b)-ab] = 0}.

Consider a^2[y(s-c)-z(s-b)] + x(b-c)[s(b+c)-bc] = 0 ……(1)

Replace x = af(A, B, C), y = bf(B, C, A), z = cf(C, A, B)

where f(A, B, C) = = a[-a^2tan A/2 + b^2tan B/2 + c^2tan C/2]

a^2[y(s-c)-z(s-b)] = a^2[b(s-c) f(B, C, A) – c(s-b) f(C, A, B)]

$= a^2$ {s[b f(B, C, A) - c f(C, A, B)] – bc [f(B,C,A) - f(C, A, B)] }

$= a^2\{a^2$tanA/2(b-c) (sb+sc-bc)–b^2 tanB/2(b-c) (sb+sc-bc)-c^2tan C/2(b-c) (sb+sc-bc)}

$=-a^2$(b-c) (sb+sc-bc) f(A, B, C) = - x(b-c)[s(b+c)-bc

Hence (1) is satisfied by the point X(197) {af(A, B, C) : bf(B, C, A) : cf(C, A, B)}
That is the line $N_A L'_A$ contains X(197).
In a similar manner, we can verify that $N_B L'_B$ and $N_C L'_C$ also contains X(197).
So, the lines $N_A L'_A$, $N_B L'_B$ and $N_C L'_C$ are concur at X(197). *(see figure-4)*

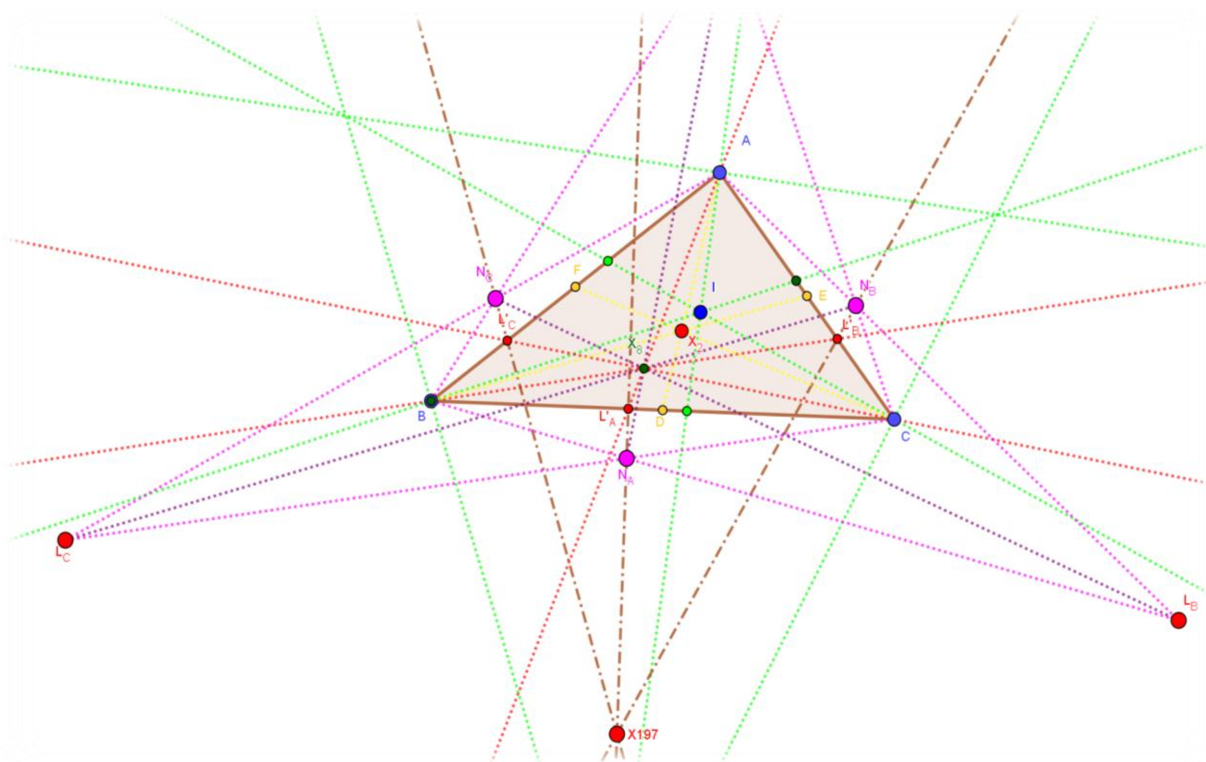

Figure 4.

Preposition -5

Carnot's theorem

Suppose a conic L intersect in the side line BC at X, X', *CA at* Y, Y' *and AB at* Z, Z' *then*

$$\left(\frac{BX}{XC}\right)\left(\frac{BX'}{X'C}\right)\left(\frac{CY}{YA}\right)\left(\frac{CY'}{Y'A}\right)\left(\frac{AZ}{ZB}\right)\left(\frac{AZ'}{Z'B}\right)=1$$

Preposition -6

If X, Y, Z are the traces of a point P and X', Y' *and* Z' *are the traces of another point Q on the sides of BC, CA, and AB of the triangle ABC such that P= (u : v: w) and Q =* $(u':v':w')$, *then using Carnot's theorem there is a conic through six points. The equation of the conic is*

$$\sum_{cyclic}\left[\frac{x^2}{uu'}-\left(\frac{1}{vw'}+\frac{1}{v'w}\right)yz\right]=0, \text{ [4].}$$

Theorem -5

a) *There exists conic through (D, E, F) and (* N'_A, N'_B, N'_C *)whose barycentric equation is*

$$\sum_{cyclic}\left[\frac{x^2}{(b+c)^2}-\left(\frac{1}{(a+b)^2}+\frac{1}{(c+a)^2}\right)yz\right]=0$$

b) *There exists conic through (* N'_A, N'_B, N'_C *) and* $\left(L'_A, L'_B, L'_C\right)$ *whose barycentric equation is*

$$\sum_{cyclic}\left[\frac{x^2}{(s-a)(b+c)^2}-\left(\frac{1}{(s-b)(a+b)^2}+\frac{1}{(s-c)(c+a)}\right)yz\right]=0$$

c) *There exists conic through (D, E, F) and* $\left(L'_A, L'_B, L'_C\right)$ *whose barycentric equation is*

$$\sum_{cyclic}\left[\frac{x^2}{(s-a)}-\left(\frac{1}{(s-c)}+\frac{1}{(s-b)}\right)yz\right]=0$$

Proof:

Clearly the triads *(D, E, F),* (N'_A, N'_B, N'_C) and $\left(L'_A, L'_B, L'_C\right)$ are the traces of centroid {X(2)} (1 : 1: 1), the isogonal conjugate of first Hatzipolakis-Yiu Point{X(594)} ($(c+b)^2$: $(c+a)^2$: $(a+b)^2$) and Nagel Point {X(8)}((s-a) : (s-b) : (s-c)).

Hence, using Carnot's theorem there exists a conic through the traces of any two of the following three points X(2), X(594) and X(8).
Their corresponding barycentric equations can be calculated using preposition 6.
Using tools similar to those used in this article, the following generalization can be demonstrated.

Generalization

In a given triangle *ABC*, $P_a(0:v:w)$, $P_b(u:0:w)$, $P_c(u:v:0)$ are the traces of an arbitrary point *P(u : v : w)* on the sides *BC, CA, AB* respectively. P_A, P_B and P_C are the reflections of the points P_a, P_b, P_c with respect to the external angular bisectors of angles *A*, *B* and *C*. Let *Q* be the isogonal conjugate of *P*. Consider the points R_A, R_B and R_C as $R_A = BP_C \cap CP_B$, $R_B = AP_C \cap CP_A$ and $R_C = BP_A \cap AP_B$, then

(I) ***The line segments*** $\boldsymbol{AR_A}$***,*** $\boldsymbol{BR_B}$ ***and*** $\boldsymbol{CR_C}$ ***concur at R.***

(II) ***The line segments*** $\boldsymbol{P_AR_A}$***,*** $\boldsymbol{P_BR_B}$ ***and*** $\boldsymbol{P_CR_C}$ ***concur at T.***
Consider the points R'_A, R'_B, R'_C, T'_A, T'_B, T'_C, M'_A, M'_B and M'_C as $R'_A = AR_A \cap BC$, $R'_B = BR_B \cap CA$, $R'_A = CR_A \cap AB$, $T'_A = AT \cap BC$, $T'_B = BT \cap CA$, $T'_C = CT \cap AB$, $S'_A = P_AR_A \cap BC$ $S'_B = P_BR_B \cap CA$ and $S'_C = P_CR_C \cap AB$ then

(III) ***The line segments*** AS'_A, BS'_B ***and*** CS'_C ***are concur at S.***

(IV) ***The lines*** $R_A T'_A$, $R_B T'_B$ ***and*** $R_C T'_C$ ***are concur at Z.***

(V) ***The points T, Q and R are collinear.***

For further study about Nagel's point (X(8)) [2] **is** referred.

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